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Uniqueness and reconstruction for the anisotropic Helmholtz decomposition

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Abstract

The Helmholtz decomposition theorem for an anisotropic medium is explicitly stated in terms of two symmetric and positive definite dyadics, which carry the directional characteristics of the medium. It is shown that this general decomposition is not unique, and that once the scalar invariant with respect to one of the dyadics and the vector invariant with respect to the other dyadic are given the initial field can be completely reconstructed.

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1. Introduction

The basic idea of representing a vector field as the gradient of a scalar plus the rotation of a vector function appears, for the first time, in a paper by Stokes [25] on diffraction in 1849. Helmholtz [11] redeveloped this decomposition in an article on the hydrodynamics of vortex motion in 1858. This result, which is known today as the Helmholtz decomposition theorem, plays an irreplaceable role in all of mathematical physics. A weak formulation of the Helmholtz decomposition is demonstrated in [4]. Furthermore, Gregory [9] has shown that the Helmholtz decomposition is true without any growth condition at infinity and he has also investigated the case where singularities are present. The Helmholtz theorem has been recently extended to polyadic (tensors of any valence) fields [6] as well as to anisotropic media [17].

In the present work the questions of uniqueness for the anisotropic decomposition, as well as that of reconstructing the field from its invariants [28], are addressed. It is proved that, if $\tilde{S}$ and $\tilde{\boldsymbol{T}}$ are the positive definite symmetric dyadics defining the S-gradient and the T-gradient, respectively, then uniqueness holds up to an additive T-gradient of a scalar ST-harmonic function. Equivalently, uniqueness holds up to an additive S-rotation of a vector ST-harmonic function. Furthermore, it is shown that once the S-divergence and the T-rotation of a vector field are given a solution of the ST-Poisson equation is all we need to reconstruct the initial field. Such a solution is also provided in terms of a quadrature, just as in the case of the isotropic

Helmholtz decomposition. Helmholtz decomposition furnishes the starting point for many differential representations, such as Stokes [12, 14, 25–27], Papkovich [3, 5, 7, 10, 20–22, 24], Neuber [10, 18], Boussinesq [10], Galerkin [8], Love [10], Palaniappan [19] and so on. Therefore, it would be of great interest to investigate what form all these representations take for fully or partially anisotropic media. The books by Serdyukov *et al* [23] and Lindell *et al* [15] provide excellent starting points for such an investigation.

Section 2 states the basic decomposition theorem for isotropic media while its anisotropic generalization is exposed in section 3. Section 4 deals with the question of uniqueness and section 5 discusses the problem of reconstructing the field from its invariants.

2. Isotropic Helmholtz theorem

The classical (isotropic) Helmholtz theorem states that, if Ω is a bounded, regular in the sense of Kellogg [13], domain in $\mathbb{R}^3$ and if $\boldsymbol{f}$ is a vector field which is continuous on $\bar{\Omega} = \Omega \cup \partial\Omega$ and has continuous first derivatives in Ω, then there are functions Φ and $\boldsymbol{A}$ defined on Ω, also with continuous first-order derivatives, such that

$$\boldsymbol{f}(\boldsymbol{r}) = \nabla\Phi(\boldsymbol{r}) + \nabla \times \boldsymbol{A}(\boldsymbol{r}) \tag{1}$$

and

$$\nabla \cdot \boldsymbol{A}(\boldsymbol{r}) = 0 \tag{2}$$

where all functions $\boldsymbol{f}$, Φ and $\boldsymbol{A}$ are defined on $\bar{\Omega}$.

The $\nabla\Phi$ part of decomposition (1) is irrotational and the $\nabla \times \boldsymbol{A}$ part is solenoidal. For exterior domains (unbounded domains with bounded boundary) with the same regularity, theorem (1), (2) is also true without any assumptions on the asymptotic decay of $\boldsymbol{f}$ at infinity, as was initially shown by Blumenthal [2] and generalized to domains that involve isolated singular points by Gregory [9].

The proof of the Helmholtz theorem is based on two remarks. First, the solution of the Poisson equation

$$\Delta\boldsymbol{u}(\boldsymbol{r}) = \boldsymbol{f}(\boldsymbol{r}) \tag{3}$$

in Ω is given by

$$\boldsymbol{u}(\boldsymbol{r}) = -\frac{1}{4\pi}\int_{\Omega}\frac{\boldsymbol{f}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}\upsilon\,(\boldsymbol{r}') \tag{4}$$

and second, any function which has continuous second-order derivatives satisfies the identity

$$\Delta\boldsymbol{u}(\boldsymbol{r}) = \nabla \otimes \nabla \cdot \boldsymbol{u}(\boldsymbol{r}) - \nabla \times (\nabla \times \boldsymbol{u}(\boldsymbol{r})). \tag{5}$$

Note that the function $\boldsymbol{u}$, given by (4), has continuous second-order derivatives because $\boldsymbol{f}$ has continuous first-order derivatives [10]. Combining (3), (4) and (5) we arrive at the conclusion that

$$\Phi(\boldsymbol{r}) = \nabla \cdot \boldsymbol{u}(\boldsymbol{r}) \tag{6}$$

and

$$\boldsymbol{A}(\boldsymbol{r}) = -\nabla \times \boldsymbol{u}(\boldsymbol{r}). \tag{7}$$

Integration by parts can then be used to write

$$\Phi(\boldsymbol{r}) = \frac{1}{4\pi}\int_{\Omega}\frac{\nabla_{r'} \cdot \boldsymbol{f}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}\upsilon\,(\boldsymbol{r}') \tag{8}$$

and

$$\boldsymbol{A}(\boldsymbol{r}) = -\frac{1}{4\pi}\int_{\Omega}\frac{\nabla_{\boldsymbol{r}'}\times\boldsymbol{f}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}\upsilon\,(\boldsymbol{r}'), \tag{9}$$

which shows that the vector field $\boldsymbol{f}$ can be reconstructed from the scalar and vector invariants of its gradient, via (1), (8) and (9).

Furthermore, the decomposition (1) is unique up to the additive gradient of a harmonic function. That is, we can always write (1) as

$$\boldsymbol{f}(\boldsymbol{r}) = [\nabla\Phi(\boldsymbol{r}) + \nabla\upsilon(\boldsymbol{r})] + [\nabla\times\boldsymbol{A}(\boldsymbol{r}) - \nabla\upsilon(\boldsymbol{r})] \tag{10}$$

where

$$\Delta\upsilon(\boldsymbol{r}) = 0 \tag{11}$$

in Ω. Obviously $\nabla\upsilon$ can contribute to the $\nabla\Phi$ part since it is irrotational by structure and it can contribute to the $\nabla\times\boldsymbol{A}$ part since, by (11), is solenoidal.

3. Anisotropic Helmholtz theorem

Let $\tilde{\boldsymbol{S}}$ be a symmetric and positive definite dyadic in $\mathbb{R}^3$ and define the S-gradient ∇_S by

$$\nabla_S = \tilde{\boldsymbol{S}}\cdot\nabla. \tag{12}$$

In fact, if s_1, s_2, s_3 are the three positive eigenvalues of $\tilde{\boldsymbol{S}}$ corresponding to the three orthogonal unit eigendirections $\hat{\boldsymbol{s}}_1, \hat{\boldsymbol{s}}_2, \hat{\boldsymbol{s}}_3$, then it is easily seen that

$$\nabla_S = \begin{bmatrix} s_1\frac{\partial}{\partial s_1} \\ s_2\frac{\partial}{\partial s_2} \\ s_3\frac{\partial}{\partial s_3} \end{bmatrix}. \tag{13}$$

Therefore, the three equivalent directions of differentiation, represented by the operator ∇, are replaced through $\tilde{\boldsymbol{S}}$ by three unequal directions of differentiation, represented by the operator ∇_S. As a result of this transformation the gradient is scaled and rotated by the standards imposed by $\tilde{\boldsymbol{S}}$. In other words, the directional characteristics of the anisotropic medium, described by $\tilde{\boldsymbol{S}}$, are incorporated within the operator ∇_S, which is now medium dependent. Obviously, a different symmetry is described by a different positive definite dyadic $\tilde{\boldsymbol{T}}$, which provides the T-gradient

$$\nabla_T = \begin{bmatrix} \tau_1\frac{\partial}{\partial \tau_1} \\ \tau_2\frac{\partial}{\partial \tau_2} \\ \tau_3\frac{\partial}{\partial \tau_3} \end{bmatrix} \tag{14}$$

caring the characteristics of $\tilde{\boldsymbol{T}}$. In a straightforward manner we define the S-divergence and the S-rotation of a vector field $\boldsymbol{f}$ by

$$\nabla_S\cdot\boldsymbol{f} = (\tilde{\boldsymbol{S}}\cdot\nabla)\cdot\boldsymbol{f} \tag{15}$$

and

$$\nabla_S\times\boldsymbol{f} = (\tilde{\boldsymbol{S}}\cdot\nabla)\times\boldsymbol{f} \tag{16}$$

respectively. Similarly, the ST-Laplacian is defined as

$$\begin{aligned}\nabla_S\cdot\nabla_T &= (\tilde{\boldsymbol{S}}\cdot\nabla)\cdot(\tilde{\boldsymbol{T}}\cdot\nabla) = \nabla\cdot\tilde{\boldsymbol{S}}\cdot\tilde{\boldsymbol{T}}\cdot\nabla = (\tilde{\boldsymbol{S}}\cdot\tilde{\boldsymbol{T}}):\nabla\otimes\nabla \\ &= \left(\sum_{i=1}^{3}s_i\hat{\boldsymbol{s}}_i\frac{\partial}{\partial s_i}\right)\cdot\left(\sum_{j=1}^{3}\tau_j\hat{\boldsymbol{\tau}}_j\frac{\partial}{\partial \tau_j}\right),\end{aligned} \tag{17}$$

where $\hat{s}_i$, $i = 1, 2, 3$, is the characteristic system associated with the dyadic $\tilde{\boldsymbol{S}}$, and $\hat{\boldsymbol{\tau}}_i$, $i = 1, 2, 3$, is the characteristic system associated with the dyadic $\tilde{\boldsymbol{T}}$.

The case $\tilde{\boldsymbol{S}} = \tilde{\boldsymbol{T}}$ implies that

$$\nabla_S \cdot \nabla_S = s_1^2 \frac{\partial^2}{\partial s_1^2} + s_2^2 \frac{\partial^2}{\partial s_2^2} + s_3^2 \frac{\partial^2}{\partial s_3^2}, \tag{18}$$

which reduces further to a multiple of the (isotropic) Laplacian whenever $s_1 = s_2 = s_3$.

The basic identity that replaces (5) reads as follows

$$(\nabla_S \cdot \nabla_T)\boldsymbol{h}(\boldsymbol{r}) = \nabla_T(\nabla_S \cdot \boldsymbol{h}(\boldsymbol{r})) - \nabla_S \times (\nabla_T \times \boldsymbol{h}(\boldsymbol{r})), \tag{19}$$

where $\boldsymbol{h}$ is any vector field with continuous second derivatives. Suppose now that $\boldsymbol{f}$ is a continuous differentiable vector field and $\boldsymbol{h}$ is a solution of the ST-Poisson equation

$$\nabla_S \cdot \nabla_T \boldsymbol{h}(\boldsymbol{r}) = \boldsymbol{f}(\boldsymbol{r}). \tag{20}$$

Then the anisotropic version of the Helmholtz decomposition theorem assumes the form

$$\boldsymbol{f}(\boldsymbol{r}) = \nabla_T \Phi(\boldsymbol{r}) + \nabla_S \times \boldsymbol{A}(\boldsymbol{r}) \tag{21}$$

where the T-gradient of Φ is a T-irrotational field and the S-rotation of $\boldsymbol{A}$ is an S-solenoidal field. The scalar potential Φ and the vector potential $\boldsymbol{A}$ are given by

$$\Phi(\boldsymbol{r}) = \nabla_S \cdot \boldsymbol{h}(\boldsymbol{r}) \tag{22}$$

and

$$\boldsymbol{A}(\boldsymbol{r}) = -\nabla_T \times \boldsymbol{h}(\boldsymbol{r}) \tag{23}$$

respectively, with $\boldsymbol{h}$ a solution of (20). Note that the T-irrotational part of $\boldsymbol{f}$ is given by an S-divergence and the S-solenoidal part of $\boldsymbol{f}$ is given by a T-rotation.

Applying the decomposition (21) to the vector potential $\boldsymbol{A}$ we obtain

$$\boldsymbol{A}(\boldsymbol{r}) = \nabla_S \Phi'(\boldsymbol{r}) + \nabla_T \times \boldsymbol{A}'(\boldsymbol{r}). \tag{24}$$

However, $\boldsymbol{A}$ enters the initial decomposition (21) of $\boldsymbol{f}$ via its S-rotation, which is

$$\nabla_S \times \boldsymbol{A}(\boldsymbol{r}) = \nabla_S \times \nabla_S \Phi'(\boldsymbol{r}) + \nabla_S \times (\nabla_T \times \boldsymbol{A}'(\boldsymbol{r})) = \nabla_S \times (\nabla_T \times \boldsymbol{A}'(\boldsymbol{r})). \tag{25}$$

Formula (25) shows that the choice of the scalar potential Φ' does not affect the decomposition (21) and therefore could be omitted, but ignoring the term $\nabla_S \Phi'$ in (24) is equivalent to assuming that $\boldsymbol{A}$ is T-solenoidal. Consequently the gauge condition assumes the form

$$\nabla_T \cdot \boldsymbol{A}(\boldsymbol{r}) = (\tilde{\boldsymbol{T}} \cdot \nabla) \cdot \boldsymbol{A} = 0. \tag{26}$$

In other words, the S-solenoidal part of $\boldsymbol{A}$ is expressed in terms of a T-solenoidal vector potential.

4. Uniqueness

Let us assume that there are two pairs of potentials (Φ, $\boldsymbol{A}$) and (Φ', $\boldsymbol{A}'$) that decompose the same vector field $\boldsymbol{f}$. Then

$$\boldsymbol{f} = \nabla_T \Phi + \nabla_S \times \boldsymbol{A}, \qquad \nabla_T \cdot \boldsymbol{A} = 0 \tag{27}$$

and

$$\boldsymbol{f} = \nabla_T \Phi' + \nabla_S \times \boldsymbol{A}', \qquad \nabla_T \cdot \boldsymbol{A}' = 0, \tag{28}$$

which imply that

$$\nabla_T(\Phi - \Phi') + \nabla_S \times (\boldsymbol{A} - \boldsymbol{A}') = \boldsymbol{0}. \tag{29}$$

Taking first the T-divergence and then the S-rotation of (29) we arrive at

$$(\nabla_S \cdot \nabla_T)(\Phi - \Phi') = 0 \tag{30}$$

and

$$\nabla_T \times [\nabla_S \times (\boldsymbol{A} - \boldsymbol{A}')] = \boldsymbol{0}. \tag{31}$$

In view of the basic identity (19), expression (31) implies that

$$\nabla_S[\nabla_T \cdot (\boldsymbol{A} - \boldsymbol{A}')] = (\nabla_T \cdot \nabla_S)(\boldsymbol{A} - \boldsymbol{A}'), \tag{32}$$

which, by (27) and (28), reduces to

$$(\nabla_T \cdot \nabla_S)(\boldsymbol{A} - \boldsymbol{A}') = \boldsymbol{0}. \tag{33}$$

From (30), (33) and the symmetry of $\tilde{\boldsymbol{S}}$ and $\tilde{\boldsymbol{T}}$ we conclude that the difference $\Phi - \Phi'$ of the scalar potentials and the difference $\boldsymbol{A} - \boldsymbol{A}'$ of the T-solenoidal vector potential are both ST-harmonic.

In other words,

$$\Phi = \Phi' + u \tag{34}$$

and

$$\boldsymbol{A} = \boldsymbol{A}' + \boldsymbol{V}, \qquad \nabla_T \cdot \boldsymbol{V} = 0, \tag{35}$$

where

$$(\nabla_S \cdot \nabla_T)u = 0 \tag{36}$$

and

$$(\nabla_S \cdot \nabla_T)\boldsymbol{V} = \boldsymbol{0}, \tag{37}$$

while (27) and (28) provide

$$\boldsymbol{f} = \nabla_T\Phi + \nabla_S \times \boldsymbol{A} = \nabla_T\Phi' + \nabla_T u + \nabla_S \times \boldsymbol{A}' + \nabla_S \times \boldsymbol{V}. \tag{38}$$

We also obtain from (35) and (37) that

$$\nabla_T \times (\nabla_S \times \boldsymbol{V}) = \nabla_S(\nabla_T \cdot \boldsymbol{V}) - (\nabla_T \cdot \nabla_S)\boldsymbol{V} = \boldsymbol{0}, \tag{39}$$

which confirms that the S-rotation of $\boldsymbol{V}$, $\nabla_S \times \boldsymbol{V}$, is T-irrotational.

Consequently, there exists a scalar field w such that

$$\nabla_S \times \boldsymbol{V} = \nabla_T w \tag{40}$$

and satisfaction of (38) demands that

$$\nabla_T(u + w) = \boldsymbol{0} \tag{41}$$

or

$$w = -u + c, \tag{42}$$

where the constant c could be assumed to be zero because only the gradient of w is needed.

We conclude that the decomposition (27) is unique up to an additive term which is the T-gradient of an ST-harmonic scalar function.

In other words,

$$\boldsymbol{f} = (\nabla_T\Phi + \nabla_T u) + (\nabla_S \times \boldsymbol{A} - \nabla_T u), \tag{43}$$

where the vector field $\nabla_T u$ is T-irrotational by structure, and it is S-solenoidal since u is ST-harmonic.

Obviously, (40) could be used the other way around and express (43) as

$$\boldsymbol{f} = (\nabla_T \Phi - \nabla_S \times \boldsymbol{V}) + (\nabla_S \times \boldsymbol{A} + \nabla_S \times \boldsymbol{V}). \tag{44}$$

Then, the uniqueness of the decomposition is secured up to an additive term which is the S-rotation of an ST-harmonic vector function. The S-rotation of $\boldsymbol{V}$ is T-irrotational because of (39) and it is S-solenoidal by structure.

For the degenerate case, where $\tilde{\boldsymbol{S}} = \tilde{\boldsymbol{T}} = \tilde{\boldsymbol{I}}$, the isotropic case (10) is recovered.

5. Decomposition and reconstruction

In this section we address the questions of how to obtain a pair of potentials Φ and $\boldsymbol{A}$ in terms of the original vector field $\boldsymbol{f}$ and how to reconstruct the field $\boldsymbol{f}$ from its S-divergence and T-rotation. To this end we take the S-divergence and the T-rotation of the representation

$$\boldsymbol{f}(\boldsymbol{r}) = \nabla_T \Phi(\boldsymbol{r}) + \nabla_S \times \boldsymbol{A}(\boldsymbol{r}) \tag{45}$$

to arrive at

$$\nabla_S \cdot \boldsymbol{f}(\boldsymbol{r}) = \nabla_S \cdot \nabla_T \Phi(\boldsymbol{r}) \tag{46}$$

$$\nabla_T \times \boldsymbol{f}(\boldsymbol{r}) = \nabla_T \times (\nabla_S \times \boldsymbol{A}(\boldsymbol{r})) = \nabla_S(\nabla_T \cdot \boldsymbol{A}(\boldsymbol{r})) - (\nabla_T \cdot \nabla_S)\boldsymbol{A}(\boldsymbol{r}) = -(\nabla_T \cdot \nabla_S)\boldsymbol{A}(\boldsymbol{r}), \tag{47}$$

where the basic identity (19) and the gauge condition (26) have been used.

Consequently, if the scalar invariant of $\nabla_S \boldsymbol{f}$ and the vector invariant of $\nabla_T \boldsymbol{f}$ are given, then the potentials Φ and $\boldsymbol{A}$ are obtained as solutions of the ST-Poisson equations

$$(\tilde{\boldsymbol{S}} \cdot \tilde{\boldsymbol{T}}) : \nabla \otimes \nabla \Phi(\boldsymbol{r}) = \nabla_S \cdot \boldsymbol{f}(\boldsymbol{r}) \tag{48}$$

and

$$(\tilde{\boldsymbol{S}} \cdot \tilde{\boldsymbol{T}}) : \nabla \otimes \nabla \boldsymbol{A}(\boldsymbol{r}) = -\nabla_T \times \boldsymbol{f}(\boldsymbol{r}). \tag{49}$$

The solutions of (48) and (49) are provided in the form

$$\Phi(\boldsymbol{r}) = \int_\Omega G(\boldsymbol{r}, \boldsymbol{r}')\nabla'_S \cdot \boldsymbol{f}(\boldsymbol{r}')\,\mathrm{d}\upsilon\,(\boldsymbol{r}') \tag{50}$$

and

$$\boldsymbol{A}(\boldsymbol{r}) = -\int_\Omega G(\boldsymbol{r}, \boldsymbol{r}')\nabla'_T \times \boldsymbol{f}(\boldsymbol{r}')\,\mathrm{d}\upsilon\,(\boldsymbol{r}'), \tag{51}$$

where the fundamental solution G satisfies the equation

$$(\tilde{\boldsymbol{S}} \cdot \tilde{\boldsymbol{T}}) : \nabla \otimes \nabla G(\boldsymbol{r}, \boldsymbol{r}') = -\delta(\boldsymbol{r} - \boldsymbol{r}') \tag{52}$$

and the primed gradient indicates differentiation with respect to the variable $\boldsymbol{r}'$.

In order to solve equation (52) we first observe that, even though the product of the two symmetric dyadics $\tilde{\boldsymbol{S}}$ and $\tilde{\boldsymbol{T}}$ is not symmetric, the double contraction of $\tilde{\boldsymbol{S}} \cdot \tilde{\boldsymbol{T}}$ and of $\tilde{\boldsymbol{T}} \cdot \tilde{\boldsymbol{S}}$ with $\nabla \otimes \nabla$ leads to the same expression. Therefore,

$$(\tilde{\boldsymbol{S}} \cdot \tilde{\boldsymbol{T}}) : \nabla \otimes \nabla = \tfrac{1}{2}(\tilde{\boldsymbol{S}} \cdot \tilde{\boldsymbol{T}} + \tilde{\boldsymbol{T}} \cdot \tilde{\boldsymbol{S}}) : \nabla \otimes \nabla. \tag{53}$$

If we denote by $\tilde{\boldsymbol{U}}$ the symmetric part of $\tilde{\boldsymbol{S}} \cdot \tilde{\boldsymbol{T}}$,

$$\tilde{\boldsymbol{U}} = \tfrac{1}{2}(\tilde{\boldsymbol{S}} \cdot \tilde{\boldsymbol{T}} + \tilde{\boldsymbol{T}} \cdot \tilde{\boldsymbol{S}}), \tag{54}$$

then it is enough to solve the equation

$$\tilde{\boldsymbol{U}} : \nabla \otimes \nabla G(\boldsymbol{r}) = -\delta(\boldsymbol{r}), \tag{55}$$

where $\tilde{\boldsymbol{U}}$ is symmetric but not necessarily positive definite. Nevertheless the positive definiteness of $\tilde{\boldsymbol{U}}$ is preserved whenever $\tilde{\boldsymbol{S}}$ and $\tilde{\boldsymbol{T}}$ commute, in which case they share a common set of eigenvectors. Then, there exists a positive definite symmetric dyadic $\tilde{\boldsymbol{V}}$ such that

$$\boldsymbol{U} = \tilde{\boldsymbol{V}} \cdot \tilde{\boldsymbol{V}} \tag{56}$$

and the transformation

$$\boldsymbol{r}_V = \tilde{\boldsymbol{V}}^{-1} \cdot \boldsymbol{r}, \tag{57}$$

which provides the connection

$$\nabla_V = \tilde{\boldsymbol{V}} \cdot \nabla, \tag{58}$$

transforms equation (55) to

$$\nabla_V^2 G(\boldsymbol{r}_V) = -\delta(\boldsymbol{r}_V). \tag{59}$$

The solution of (59) is given by [16]

$$\begin{aligned} G(\boldsymbol{r}_V) &= \frac{1}{4\pi(\det \tilde{\boldsymbol{V}})|\boldsymbol{r}_V|} = \frac{1}{4\pi(\det \tilde{\boldsymbol{V}})\sqrt{(\boldsymbol{V}^{-1} \cdot \boldsymbol{r}) \cdot (\boldsymbol{V}^{-1} \cdot \boldsymbol{r})}} \\ &= \frac{1}{4\pi(\det \tilde{\boldsymbol{V}})\sqrt{\tilde{\boldsymbol{U}}^{-1} : \boldsymbol{r} \otimes \boldsymbol{r}}}, \end{aligned} \tag{60}$$

which implies that the solution of (52) is given by

$$G(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{4\pi(\det \tilde{\boldsymbol{V}})\sqrt{\tilde{\boldsymbol{U}}^{-1} : (\boldsymbol{r} - \boldsymbol{r}') \otimes (\boldsymbol{r} - \boldsymbol{r}')}}. \tag{61}$$

The fundamental solution G is also written as

$$G(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{4\pi\sqrt{\tilde{\boldsymbol{U}}^{(2)} : (\boldsymbol{r} - \boldsymbol{r}') \otimes (\boldsymbol{r} - \boldsymbol{r}')}}, \tag{62}$$

where the symmetric dyadic $\tilde{\boldsymbol{U}}^{(2)}$ represents the dyadic invariant of $\tilde{\boldsymbol{U}}$. We recall here that if

$$\tilde{\boldsymbol{A}} = \boldsymbol{a}_1 \otimes \boldsymbol{b}_1 + \boldsymbol{a}_2 \otimes \boldsymbol{b}_2 + \boldsymbol{a}_3 \otimes \boldsymbol{b}_3 \tag{63}$$

the dyadic invariant [1, 14] of $\tilde{\boldsymbol{A}}$ is given by

$$\tilde{\boldsymbol{A}}^{(2)} = (\boldsymbol{a}_1 \times \boldsymbol{a}_2) \otimes (\boldsymbol{b}_1 \times \boldsymbol{b}_2) + (\boldsymbol{a}_2 \times \boldsymbol{a}_3) \otimes (\boldsymbol{b}_2 \times \boldsymbol{b}_3) + (\boldsymbol{a}_3 \times \boldsymbol{a}_1) \otimes (\boldsymbol{b}_3 \times \boldsymbol{b}_1). \tag{64}$$

Substituting (61), or (62) into the integrals (50) and (51) we obtain the potentials Φ and $\boldsymbol{A}$ in terms of the S-divergence and T-rotation of the vector field $\boldsymbol{f}$, respectively.

Finally, once Φ and $\boldsymbol{A}$ are known the original field $\boldsymbol{f}$ is obtained through (45).

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